Bulk metric of brane world models and submanifolds in 6D pseudo-Euclidean space-time

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arXiv:1010.5949v2 [gr-qc] 3 Nov 2010

Abstract

In this short note, five-dimensional brane world models with dS_4 metric on the branes are discussed. The explicit coordinate transformations, which show the equivalence between the bulk metric of these brane world models and the metric induced on an appropriate submanifolds in the flat six-dimensional pseudo-Euclidean space-time, are presented. The cases of the zero and non-zero cosmological constant in the bulk are discussed in detail.

It is well known that the space-time of the Randall-Sundrum model [1] is a slice of the maximally-symmetric AdS_5 space-time. It means that the bulk metric of this model corresponds to the metric induced on a hyperboloid embedded in a six-dimensional pseudo-Euclidean space-time. Moreover, it was shown in [2] that the bulk metric of the form

$$ds^2 = A^2(y)\left(-dt^2 + \eta_{ij}dx^i dx^j\right) + dy^2, \tag{1}$$

where $\eta_{ij} = diag(1,1,1)$, $i,j = 1,2,3$, can be represented as the metric induced on the hypersurface

$$T^2 - \eta_{ij}X^iX^j - Y^2 + Z^2 = \left[-A(y)\int \frac{dy}{dA/dy}\right]_{y=A^{-1}\left(\frac{2(Z-Y)}{\alpha}\right)}, \tag{2}$$

where $\alpha \neq 0$ is an arbitrary constant, in six-dimensional pseudo-Euclidean space-time with the metric

$$ds^2 = -dT^2 + \eta_{ij}dX^idX^j + dY^2 - dZ^2 \tag{3}$$

provided that

$$\begin{aligned}
Y &= \frac{1}{\alpha}\left(A(y)\left(-t^2 + \eta_{ij}x^ix^j\right) - \int \frac{dy}{dA/dy}\right) - \alpha\frac{A(y)}{4}, && (4)\\
Z &= \frac{1}{\alpha}\left(A(y)\left(-t^2 + \eta_{ij}x^ix^j\right) - \int \frac{dy}{dA/dy}\right) + \alpha\frac{A(y)}{4}, && (5)\\
T &= A(y)t, && (6)\\
X^i &= A(y)x^i. && (7)
\end{aligned}$$

For the Randall-Sundrum model with $A = e^{-ky}$ (for simplicity we use y instead of $|y|$) we obtain the well-known result

$$T^2 + Z^2 - \eta_{ij}X^iX^j - Y^2 = \frac{1}{k^2}. \tag{8}$$

Meanwhile there is a large class of brane world models with dS_4 metric on the branes, some of them can be found in [3, 4]. The bulk metric of such models has the form

$$ds^2 = A^2(y)\left(-dt^2 + e^{2\lambda t}\eta_{ij}dx^idx^j\right) + dy^2 \tag{9}$$

and it seems to be interesting to find an analogous hypersurface for such form of the metric.

To solve this problem first let us consider a much simpler case – the case of the flat five-dimensional pseudo-Euclidean space-time with the metric

$$ds^2_{(5)} = -dT^2 + \eta_{ij} dX^i dX^j + dY^2. \tag{10}$$

With the help of coordinate transformations

$$\begin{aligned} T &= \alpha\lambda y \left(\eta_{ij} x^i x^j e^{\lambda t} - \frac{e^{-\lambda t}}{\lambda^2} \right) + y \frac{\lambda}{4\alpha} e^{\lambda t}, && (11) \\ Y &= \alpha\lambda y \left(\eta_{ij} x^i x^j e^{\lambda t} - \frac{e^{-\lambda t}}{\lambda^2} \right) - y \frac{\lambda}{4\alpha} e^{\lambda t}, && (12) \\ X^i &= \lambda y x^i e^{\lambda t}, && (13) \end{aligned}$$

where $\alpha \neq 0$ is a constant, we can get from (10)

$$ds^2 = \lambda^2 y^2 \left(-dt^2 + e^{2\lambda t} \eta_{ij} dx^i dx^j \right) + dy^2 \tag{14}$$

(note that metric (14) is very similar to the metric of the well-known Milne universe [5], see the Appendix).

An important point is that metric (14) in the bulk admits the existence of two branes with negative and positive tensions (and with different absolute values of the tensions), which can be located at the points $0 < y_1 < y_2$ of the orbifold with $y_1 \leq y \leq y_2$ for $y > 0$ and points $-y$ and y identified. Indeed, it follows from the Einstein equations (see, for example, [3]) that the brane tensions should be such that $\epsilon_1 \sim -\frac{A'}{A}|_{y=y_1} = -\frac{1}{y_1}$ and $\epsilon_2 \sim \frac{A'}{A}|_{y=y_2} = \frac{1}{y_2}$. Thus, even the empty bulk supports the existence of branes, moreover, the branes appear to be inflating in this case (see examples of such brane world models with empty five-dimensional bulk and discussion of their properties in [6]).

Now with the help of formulas (11)-(13) one can easily obtain the equation for the submanifold which corresponds to metric (9) (below we will not take into account the existence of branes and concentrate on the metric in the bulk). Indeed, let us consider six-dimensional pseudo-Euclidean space-time with the metric

$$ds^2_{(6)} = -dT^2 + \eta_{ij} dX^i dX^j + dY^2 \mp dZ^2 \tag{15}$$

and the coordinate transformations

$$\begin{aligned} T &= \alpha\lambda F(y) \left(\eta_{ij} x^i x^j e^{\lambda t} - \frac{e^{-\lambda t}}{\lambda^2} \right) + F(y) \frac{\lambda}{4\alpha} e^{\lambda t}, && (16) \\ Y &= \alpha\lambda F(y) \left(\eta_{ij} x^i x^j e^{\lambda t} - \frac{e^{-\lambda t}}{\lambda^2} \right) - F(y) \frac{\lambda}{4\alpha} e^{\lambda t}, && (17) \\ X^i &= \lambda F(y) x^i e^{\lambda t}, && (18) \\ Z &= G(y). && (19) \end{aligned}$$

One can see that (16)-(18) are simply (11)-(13) with $y \to F(y)$. Substituting (16)-(19) into (15) we get

$$ds^2 = \lambda^2 F^2(y) \left(-dt^2 + e^{2\lambda t} \eta_{ij} dx^i dx^j \right) + \left(F'^2(y) \mp G'^2(y) \right) dy^2, \tag{20}$$

where $' = \frac{d}{dy}$. So with

$$F(y) = \frac{A(y)}{\lambda} \tag{21}$$

and

$$F'^2 \mp G'^2 = 1 \tag{22}$$

we get (9). If $F' \geq 1$, then one should take time-like extra dimension with the coordinate Z in (15) and $F'^2 - G'^2 = 1$; if $F' \leq 1$, then one should take space-like extra dimension with the coordinate Z in (15) and $F'^2 + G'^2 = 1$. The equation of the submanifold, which corresponds to (16)-(19), can be easily obtained and has the form

$$T^2 - Y^2 - \eta_{ij} X^i X^j = -F^2(y) \left.\right|_{y=G^{-1}(Z)} , \tag{23}$$

where $F(y)$ and $G(y)$ satisfy equations (21) and (22). This submanifold is embedded into (4+2) or (5+1) pseudo-Euclidean space-time depending on the form of $A(y)$.

Now let us turn to specific examples. First, we consider the brane world model, which corresponds to the negative value of the bulk cosmological constant [3], but, contrary to the case of the Randall-Sundrum model [1], provides dS_4 metric on the branes. In this model the metric in the bulk has the form

$$ds^2 = \frac{\lambda^2}{k^2} \sinh^2(ky) \left(-dt^2 + e^{2\lambda t} \eta_{ij} dx^i dx^j\right) + dy^2 \tag{24}$$

(note that metric (24) has a simpler form than that of [3], but the metric of [3] can be brought to form (24) by a simple redefinition of the parameters and a shift of the extra dimension coordinate). The value of the four-dimensional Hubble parameter λ is defined by the boundary conditions on the branes, which are not presented here. From (21) we get $F(y) = \frac{\sinh(ky)}{k}$. Thus, we should take time-like extra dimension Z and $G(y) = \frac{\cosh(ky)}{k}$. Then we easily get from (23) hyperboloid (8) embedded in (4+2) pseudo-Euclidean space-time. This is the expected result because metric (24) corresponds to the same matter in the bulk as the metric of the Randall-Sundrum model

$$ds^2 = e^{-2k\tilde{y}} \left(-dt^2 + \eta_{ij} dx^i dx^j\right) + d\tilde{y}^2. \tag{25}$$

Indeed, the direct coordinate transformations between metric (25) and metric (24) are

$$\tilde{y} = -\frac{1}{k} \ln \left(\pm \frac{\lambda}{k} \sinh(ky) e^{\lambda t} \right) , \tag{26}$$

$$\tilde{t} = \pm \frac{e^{-\lambda t}}{\lambda} \coth(ky). \tag{27}$$

One can check that substitution of coordinate transformations (26), (27) into metric (25) leads to (24). But note that although solutions (25) and (24) can be transformed one to another in the bulk, the models of [1] and [3] correspond to different physical systems because of the different relations between the values of the brane tensions used in these models.

Now we turn to the second model, which was also discussed in [3]. It describes a slice of the maximally-symmetric dS_5 space-time (because of the positive value of the cosmological constant in the bulk) and also provides dS_4 metric on the branes. The metric of the model has the form

$$ds^2 = \frac{\lambda^2}{k^2} \sin^2(ky) \left(-dt^2 + e^{2\lambda t} \eta_{ij} dx^i dx^j\right) + dy^2. \tag{28}$$

((28) also has a simpler form than that of [3], but again up to a simple redefinition of the parameters and a shift of the extra dimension's coordinate). Now we should take $F(y) = \frac{\sin(ky)}{k}$, space-like extra dimension Z and $G(y) = \frac{\cos(ky)}{k}$. Then we get from (23) the hyperboloid

$$T^2 - Z^2 - \eta_{ij}X^iX^j - Y^2 = -\frac{1}{k^2}, \tag{29}$$

embedded in (5+1) pseudo-Euclidean space-time. Of course, this is also the expected result.

We hope that the results presented in this note can be useful from theoretical and pedagogical points of view.

Acknowledgments

The author is grateful to I.P. Volobuev for discussions. The work was supported by grant of Russian Ministry of Education and Science NS-4142.2010.2, RFBR grant 08-02-92499-CNRSL_a and state contract 02.740.11.0244.

Appendix: Milne universe

Metric of the Milne universe has the form

$$ds^2 = -dt^2 + \lambda^2 t^2 \left((dx^1)^2 + e^{2\lambda x^1}\left((dx^2)^2 + (dx^3)^2\right)\right). \tag{30}$$

With the help of transformations

$$x^1 = \frac{1}{\lambda}\ln\left(\frac{2\alpha}{\lambda}\left(\cosh(\chi) - \sinh(\chi)\cos(\theta)\right)\right), \tag{31}$$

$$x^2 = \frac{1}{2\alpha}\left(\frac{\sinh(\chi)\sin(\theta)\cos(\varphi)}{\cosh(\chi) - \sinh(\chi)\cos(\theta)}\right), \tag{32}$$

$$x^3 = \frac{1}{2\alpha}\left(\frac{\sinh(\chi)\sin(\theta)\sin(\varphi)}{\cosh(\chi) - \sinh(\chi)\cos(\theta)}\right), \tag{33}$$

where α is a constant, metric (30) can be brought to the more familiar form (see, for example, "expanding Minkowski universe" in [7])

$$ds^2 = -dt^2 + t^2\left(d\chi^2 + \sinh^2(\chi)\left(d\theta^2 + \sin^2(\theta)d\varphi^2\right)\right). \tag{34}$$

References

[1] L. Randall and R. Sundrum, *Phys. Rev. Lett.* **83** (1999) 3370.

[2] M.N. Smolyakov, *Class. Quant. Grav.* **25** (2008) 238003; Erratum-ibid. **27** (2010) 109801.

[3] N. Kaloper, *Phys. Rev. D* **60** (1999) 123506.

[4] A. Karch and L. Randall, *JHEP* **0105** (2001) 008.
J.M. Cline and H. Firouzjahi, *Phys. Lett. B* **514** (2001) 205.
P. Kanti, S.c. Lee and K.A. Olive, *Phys. Rev. D* **67** (2003) 024037.

[5] E.A. Milne, *Nature* **130** (1932) 9.

[6] P. Binetruy, C. Deffayet and D. Langlois, *Nucl. Phys. B* **565** (2000) 269.
N. Deruelle and T. Dolezel, *Phys. Rev. D* **62** (2000) 103502.

[7] C.W. Misner, K.S. Thorne and J.A. Wheeler, *Gravitation* (W.H. Freeman and Company, San Francisco, 1973), pp. 743-744.